Field of a Uniformly Moving Charge in an Anisotropic Dispersive Medium

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The electromagnetic field of a point charge moving uniformly in a uniaxial dispersive medium is studied in the rest frame of the charge. It is shown that the Fourier integral for the scalar potential breaks up into three integrals, two of which are formally identical to the isotropic integral and yield the ordinary and extraordinary cones. Using the convolution theorem of the Fourier transform, the third integral is reduced to an integral over the isotropic field. Dispersion is explicitly introduced into the problem and the isotropic field is evaluated on the basis of a simplified dispersion formula. The effect of dispersion on the field cone is studied as a function of the cut-off frequency.

1. Introduction

In this paper, the well-known field of a point charge moving uniformly in an isotropic medium [1–4] is generalized, taking account of dispersion and anisotropy. The field is calculated in an inertial frame at rest with the charge, where both E and H can be derived from scalar potentials φ and ψ respectively. The electric potential φ is worked out here, and ψ can be obtained from considerations of symmetry of the field equations. Using the modified constitutive relations in this frame [5] and the electrodynamic transformation equations, one can derive the complete electromagnetic field in the rest frame of the medium.

Motion of the charge perpendicular to the optic axis of an unbounded uniaxial dielectric is dealt with here. The techniques of residue calculus enable us to evaluate the Fourier integral for the scalar potential φ. It is shown that the angular distribution of the field within the cones can be calculated, using the convolution properties of the Fourier transform. The development presented here has the advantage that it is amenable to the explicit introduction of dispersion into the problem. Dispersion not only provides a cut-off for the energy radiated but also diffuses the sharp cone of infinite field. Field now penetrates the barrier presented earlier by the surface of the cone and emerges ahead of the charge. The magnitude of such effects is studied as a function of the cut-off frequency and the oscillatory nature of the resulting field is depicted graphically.

Applying a set of transformations, the results of this paper can be extended easily to include motion in an arbitrary direction in a doubly anisotropic medium. The medium can even have non-coincident principal axes of permittivity and permeability provided it satisfies a generalised uniaxiality condition [6].

2. Field in an Anisotropic Medium

Let us choose the x_1 axis along the direction of motion and the x_1-x_2 plane as a principal plane of the medium, denoting the angle between x_1 and the optic axis by Ω. In the rest frame of the charge, the Fourier integral for the scalar potential is given by [5]

$$\varphi = \frac{e}{8\pi^3}\left[\int_{-\infty}^{\infty} \frac{r^2(1-\beta^2\varepsilon_{22})\, e^{ik\cdot x}\, d^3k}{r^2(\varepsilon_{11}-\beta^2\varepsilon_1\varepsilon_2)\,k_1^2+\varepsilon_{22}k_2^2+\varepsilon_2 k_3^2+2r\varepsilon_{12}k_1k_2} + \beta^2 r^2 \sin^2\Omega\,(\varepsilon_1-\varepsilon_2)\right.$$
$$\left.\times\int_{-\infty}^{\infty}\frac{k_3^2\, e^{ik\cdot x}\, d^3k}{[r^2(1-\beta^2\varepsilon_2)\,k_1^2+k_2^2+k_3^2]\,[r^2(\varepsilon_{11}-\beta^2\varepsilon_1\varepsilon_2)\,k_1^2+\varepsilon_{22}k_2^2+\varepsilon_2k_3^2+2r\varepsilon_{12}k_1k_2]}\right] \quad (1)$$

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where $\varepsilon_2 = \varepsilon_3 + \varepsilon_1$. The salient features of the field in the general case of motion of the charge in an arbitrary direction make their appearance even when the particle moves perpendicular to the optic axis. We therefore study this special case, for which the scalar potential reduces to

$$\varphi = \frac{e}{8\pi^3}\left[-\int_{-\infty}^{\infty} \frac{\alpha_1^2}{\varepsilon_2} \frac{e^{ik\cdot x}}{k_3^2 - k_0'^2} \mathrm{d}^3 k + \beta^2 r^2 (\varepsilon_1 - \varepsilon_2) \int_{-\infty}^{\infty} \frac{k_3^2 e^{ik\cdot x} \mathrm{d}^3 k}{\varepsilon_2 (k_3^2 - k_0^2)(k_3^2 - k_0'^2)}\right] \tag{2}$$

where $k_0^2 = \alpha_2^2 k_1^2 - k_2^2$, $k_0'^2 = \alpha_1^2 k_1^2 - \Lambda^2 k_2^2$, $\alpha_1^2 = r^2(\beta^2 \varepsilon_1 - 1)$, $\alpha_2^2 = r^2(\beta^2 \varepsilon_2 - 1)$

and $\Lambda^2 = \varepsilon_1/\varepsilon_2$.

Breaking the second integral in (2) into partial fractions, we obtain

$$\varphi = (e/8\pi^3)\Bigg[\int [r^2(1 - \beta^2 \varepsilon_1)/\varepsilon_2]\, e^{ik_1x_1} \mathrm{d}k_1 \int e^{ik_2x_2} g_2(k_2)\, \mathrm{d}k_2$$

$$+ \beta^2 r^2 (\varepsilon_1 - \varepsilon_2) \int \frac{e^{ik_1x_1}}{\varepsilon_2} \mathrm{d}k_1 \int \frac{e^{ik_2x_2}}{\sigma^2} \left\{ g_2(k_2)\left[\Lambda^2 + \frac{r^2 k_1^2}{k_2^2 - p^2 k_1^2}\right] - g_1(k_2)\left[1 + \frac{r^2 k_1^2}{k_2^2 - p^2 k_1^2}\right]\right\} \mathrm{d}k_2\Bigg] \tag{3}$$

where $g_1(k_2) = \int \frac{e^{ik_3x_3}}{k_3^2 - k_0^2} \mathrm{d}k_3$, $g_2(k_2) = \int \frac{e^{ik_3x_3}}{k_3^2 - k_0'^2} \mathrm{d}k_3$, $\sigma^2 = \Lambda^2 - 1$, $p^2 = r^2 \beta^2 \varepsilon_2$.

$g_1(k_2)$ and $g_2(k_2)$ are formally identical to $g(k_2)$ in the isotropic case,

$$g(k_2) = \int_{-\infty}^{\infty} e^{ik_3x_3} \mathrm{d}k_3 / (k_3^2 + k_2^2 - \alpha^2 k_1^2) \qquad \text{with} \quad \alpha^2 = r^2(\beta^2 \varepsilon - 1) \tag{4}$$

which can be evaluated by the use of the residue theorem. For velocities of charge below the threshold of radiation, the poles of the k_3-integration are always imaginary. The Fourier transform of the residue at these poles gives us the value of the k_2 integral. The potential φ in an isotropic medium under these conditions is given by

$$\varphi = (e/4\pi^2) \int_{-\infty}^{\infty} (\lambda^2/\varepsilon)\, e^{ik_1x_1} K_0(\lambda |k_1| P)\, \mathrm{d}k_1 \qquad \text{where} \quad P = \sqrt{x_2^2 + x_3^2}, \ \lambda^2 = -\alpha^2. \tag{5}$$

For $\beta = 0$, Eq. (3) gives us the well-known potential of a static charge in a uniaxial medium,

$$\varphi = e/(4\pi \sqrt{\varepsilon_2} \sqrt{\varepsilon_2 x_1^2 + \varepsilon_1 P^2}). \tag{6}$$

Under radiating conditions, the poles in the k_3 integration of (4) are imaginary or real, depending on whether k_2^2 is greater or less than $\alpha^2 k_1^2$. Since k_1 is related to the frequency by $\omega = r\beta c|k_1|$, the imaginary part of ε is positive or negative according as k_1 is positive or negative. The imaginary part of k_3 is then positive at the positive pole and negative at the negative pole for positive values of k_1. The contour of integration along the real k_3 axis must therefore pass above the negative pole and below the positive pole. The situation is reversed for negative values of k_1. Depending on the sign of x_3, one or the other of these poles is included within the contour as it is closed on the upper or lower infinite semi-circle. Thus

$$g(k_2) = \pi i \left[\begin{array}{ll} \operatorname{sgn} k_1 \cdot \exp\{\operatorname{sgn} k_1\, i\, |x_3| \sqrt{\alpha^2 k_1^2 - k_2^2}\}/\sqrt{\alpha^2 k_1^2 - k_2^2} & \text{for} \quad k_2^2 < \alpha^2 k_1^2 \\ -i \cdot \exp\{-|x_3| \sqrt{k_2^2 - \alpha^2 k_1^2}\}/\sqrt{k_2^2 - \alpha^2 k_1^2} & \text{for} \quad k_2^2 > \alpha^2 k_1^2 \end{array}\right]. \tag{7}$$

The k_2 integral is the Fourier transform of this function, and

$$\varphi = -(i e/8\pi) \int_{-\infty}^{\infty} (\alpha^2/\varepsilon)\, e^{ik_1x_1} [\operatorname{sgn} k_1 \cdot J_0(\alpha |k_1| P) + i N_0(\alpha |k_1| P)]\, \mathrm{d}k_1. \tag{8}$$

In the non-dispersive case, this integral gives rise to the isotropic cone, and Eq. (3) reduces to

$$\varphi = \frac{e}{8\pi^3}\Bigg[\frac{r^2}{\varepsilon_2} \int e^{ik_1x_1} \int e^{ik_2x_2} g_2(k_2)\, \mathrm{d}k_2\, \mathrm{d}k_1 - \beta^2 r^2 \int e^{ik_1x_1} \int e^{ik_2x_2} g_1(k_2)\, \mathrm{d}k_2\, \mathrm{d}k_1 \tag{9 a, b}$$

$$+ \beta^2 r^4 \int k_1^2 e^{ik_1x_1} \int e^{ik_2x_2} \{g_2(k_2) - g_1(k_2)\}\, h(k_2)\, \mathrm{d}k_2\, \mathrm{d}k_1\Bigg] \tag{9 c}$$

where $h(k_2) = 1/(k_2^2 - p^2 k_1^2)$.

The first two integrals in (9) are formally identical to (8) and yield the ordinary and extraordinary cones, whose equations in the rest frame of the medium are

$$\left.\begin{aligned} -x_1^0+(\beta^2\varepsilon_2-1)(x_2^0+x_3^0)=0\,,\\ -\varepsilon_1 x_1^0+(\beta^2\varepsilon_2-1)(\varepsilon_2 x_2^0+\varepsilon_1 x_3^0)=0\,.\end{aligned}\right\} \qquad (10)$$

Each term in the integral (9 c) is of the form

$$\int_{-\infty}^{\infty} g(k_2)\,h(k_2)\,e^{ik_2x_2}\,\mathrm{d}k_2\,. \qquad (11)$$

We can now apply the convolution theorem to evaluate the Fourier transform of the product of $g(k_2)$ and $h(k_2)$. $h(k_2)$ has two simple poles on the real k_2 axis. The contour for this k_2 integration is once again identical to that for the k_3 integration in $g(k_2)$, with one or other of the two poles enclosed within the contour, according to the sign of k_1 and x_2. The final result of this integration is however independent of the sign of k_1 and is given as

$$\int_{-\infty}^{\infty}\exp\{i\,k_2\,x_2\}\,h(k_2)\,\mathrm{d}k_2=\pi\,i\cdot\exp\{i\,p\,k_1\,|\,x_2\,|\}/p\,k_1\,. \qquad (12)$$

Thus, the integral (9 c) is made up of terms like

$$\int_{-\infty}^{\infty}\Big\{\int_0^{\infty}k_1\cdot\exp\{i\,k_1(x_1+p\,t)\}\,[\operatorname{sgn}k_1\cdot J_0(a\,|\,k_1\,|\sqrt{(x_2\mp t)^2+x_3^2}) + i\,N_0(a\,|\,k_1\,|\sqrt{(x_2\mp t)^2+x_3^2})]\,\mathrm{d}t\Big\}\,\mathrm{d}k_1 \qquad (13)$$

where t is the supplementary variable in the Faltung. We can now interchange the order of integration, integrating first over k_1 and then over t. Thus the integrals in (9 c) are also brought into the form (8) that gave rise to the isotropic cone. (9 c) therefore vanishes whenever $x_1+p\,t$ is positive. For field points ahead of the charge the contribution of these terms is zero. For negative values of x_1, the integration over t extends only upto values within $|\,x_1\,|/p$. The energy radiated per unit path length is given by $e(\partial\varphi/\partial x_1)_{x\to 0}$. Since the upper limit of t integration involves x_1, we conclude that the terms in (9 c) do not contribute to the radiated energy, which is obtained from (9 a) and (9 b) as

$$W=(e^2/4\,\pi\,c^2)\int[1-(1/\beta^2\,\varepsilon_2)\sqrt{\varepsilon_2/\varepsilon_1}]\;\omega\,\mathrm{d}\omega\,. \qquad (14)$$

The only effect of the additional integral (9 c) is thus to distort the field distribution within the cones represented by (9 a) and (9 b).

From this discussion it follows that the calculation of the field in a uniaxial medium reduces to an integration of the field in an isotropic medium. In the non-dispersive case, the isotropic field is well-known. In presence of dispersion, the integrand is the field in an isotropic dispersive medium; and this is developed in the next section. The results of this final integration will be reported in our next paper.

3. Field in a Dispersive Medium

As a result of dispersion, the radiation condition fails to be satisfied above a certain cut-off frequency determined by the dispersion curve of the medium and the velocity of the charge. The integrals over k_1 are now truncated and are of the type (5) upto a value k_1^0, above which they take the form (8). In this section, we confine our attention to an isotropic medium.

The dielectric constant of a typical medium such as is used in Cherenkov detectors exhibits a series of abrupt drops in microwave, infrared, and ultraviolet regions corresponding to the disappearance of dipolar, ionic and orbital contributions respectively. Apart from the corresponding narrow absorption bands, the overall dispersion curve can be approximated by a series of step functions starting from a large static dielectric constant to a final value of unity at the limit of infinite frequency [7].

Let us consider one such step, with ε constant upto a certain frequency ω^0 above which it is equal to unity. The charge is assumed to satisfy the radiation condition $\beta^2\,\varepsilon>1$. The value of k_1^0 at cut-off is of the order of 10^6. We shall denote the integral (8) with $|\,k_1\,|$ running upto k_1^0 by φ_{low} and the integral (5) with $|\,k_1\,|$ above k_1^0 as φ_{vac}. We thus have, everywhere,

$$\varphi_{\text{vac}}=(e/2\,\pi^2)\int_{k_1^0}^{\infty}K_0(p\,k_1)\cos k_1\,x_1\,\mathrm{d}k_1\,. \qquad (15)$$

The argument of the modified Bessel function between the limits of integration is so large that one can safely use the leading term in the asymptotic expansion of $K_0(P\,k_1)$ down to distances of order 10^{-5} cm from the line of motion. The above integral then yields

$$\varphi_{\text{vac}}=\frac{e}{2\,\pi^2}\,\frac{P\cos x_1\,k_1^0-x_1\sin x_1\,k_1^0}{R^2}\,K_0(P\,k_1^0) \qquad (16)$$

where $R^2 = x_1^2 + x_2^2 + x_3^2$. At all distances of interest, $P\,k_1^0$ is of the order of 10^6 and the contribution of φ_{vac} to the total potential is vanishingly small.

φ_{low} can be obtained by subtracting the integral (8) between the limits k_1^0 and ∞ from the known non-dispersive potential. Let us first consider the backward field within the non-dispersive cone, where $x_1^2 > a^2 P^2$. The potential in this region is given by

$$\varphi_{\text{in}} = Q\left[\frac{2}{\sqrt{x^2 - a^2 P^2}} - \left\{\int_{k_1^0}^{\infty} [\sin k_1 x\, J_0(a\, k_1 P) - \cos k_1 x\, N_0(a\, k_1 P)]\, \mathrm{d}k_1\right\}\right] \quad (17)$$

with $Q = -e\,a^2/4\,\pi\,\varepsilon$ and $x = |x_1|$. Using the asymptotic expansions of J_0 and N_0, the potential is obtained in terms of the Fresnel integrals C and S as

$$\varphi_{\text{in}} = Q\left[\frac{2}{\sqrt{\eta\,(x + a\,P)}} - \sqrt{\frac{2}{a\,P\,\eta}}\,\{1 - [C(\sqrt{z}) + S(\sqrt{z})]\}\right] \quad (18)$$

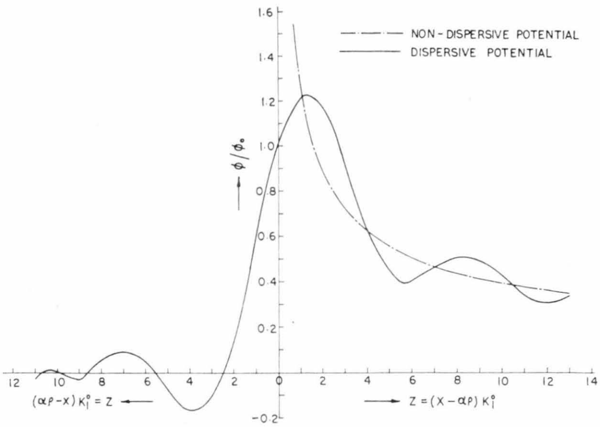
where $z = \eta\, k_1^0$; and $\eta = |x - a\,P|$ denotes the distance of the field point from the cone $x = a\,P$, measured normal to the line of motion of the charge. The most interesting region of the backward field is the vicinity of the cone where η is small and $x \cong a\,P$. Near the cone, we thus have

$$\varphi_{\text{in}} = \sqrt{\frac{\pi}{2}}\,\varphi_0\left[\frac{C(\sqrt{z}) + S(\sqrt{z})}{\sqrt{z}}\right] \quad (19)$$

where $\varphi_0 = \sqrt{2/\pi}\cdot Q\sqrt{2\,k_1^0/x}$ is the value of the potential φ_{in} at the site of the non-dispersive cone $\eta = 0$. The field outside the cone is likewise given by

$$\varphi_{\text{out}} = \sqrt{\pi/2}\,\varphi_0[(C(\sqrt{z}) - S(\sqrt{z}))/\sqrt{z}]\,. \quad (20)$$

In Fig. 1 a plot of these potentials in units of φ_0 is shown against z, together with the potential in the non-dispersive case. The sharp infinite cone has now degenerated into a conical sheath of finite field. As we proceed towards the cone from within, the potential oscillates, attains a maximum and diffuses into regions which were field-free earlier, registering an oscillatory decay there. The maximum potential is proportional to the square-root of the cut-off frequency. Instead of maintaining an infinite



value indefinitely backwards, the potential dies down inversely as the square-root of the distance behind the charge. The finite thickness of the wave-front gives rise to a finite time duration for the Cherenkov pulses which ought to be infinitely sharp in the non-dispersive case. An estimate of this thickness is given by the inward shift of the maximum of the potential. This can be obtained by a graphical solution of the equation

$$z[J_{-1/2}(z) \pm J_{1/2}(z)] = C(\sqrt{z}) \pm S(\sqrt{z}) \quad (21)$$

satisfied by the maxima and minima of the potential on either side of the cone, and is found to correspond to a distance of the order of the cut-off wavelength ($\eta\, k_1^0 = 1.285$). Moreover, the value of the dispersive potential at its maximum is nearly of the order of the non-dispersive potential at the point to which the maximum is now displaced.

The forward field also exhibits a damped oscillatory character and is similarly given by

$$\varphi_{\text{for}} = -\sqrt{\pi/2}\,\varphi^0[(C(\sqrt{y}) - S(\sqrt{y}))/\sqrt{y}] \quad (22)$$

where $y = (x_1 + a\,P)\,k_1^0$

and $\varphi^0 = \sqrt{2/\pi}\cdot Q\,\sqrt{2\,k_1^0/a\,P}\,.$

It is clear that the effect on the field, of a number of steps in the dispersion curve is additive and the general features of the field are basically unaltered. Each step gives rise to its corresponding sheath of wave-front. Even the effect of absorption bands can be dealt with approximately by a slight modification of the above model.

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